ASYMPTOTIC ANALYSIS OF MULTIPLE DESCRIPTION LATTICE VECTOR QUANTIZATION

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ABSTRACT

Recent results have shown that general K-channel multiple-description-coding (MDC) approaches often have significant advantages over conventional two-channel MDC methods. We provide new asymptotic results to describe the performance of a general K-channel symmetric MDC lattice vector quantizer (MDLVQ). We consider a memoryless L-dimensional source with probability density function f and differential entropy $h(f) <\infty$. We control the redundancy with a parameter $a \in (0,1)$ and consider a symmetric MDC with a K-tuple of $\{R, R, \cdots, R\}$ as side quantizer rates. We show that if κ out of K descriptions are received, then the central distortion $D^{(K,K)}$ and the side distortions $D^{(K,\kappa)}$ satisfy

$$\lim_{R\to\infty} D^{(K,K)} 2^{2R[1+a(K-1)]} = G(\Lambda)2^{2h(f)},$$

$$\lim_{R\to\infty} D^{(K,\kappa)} 2^{2R(1-a)} = C(K,\kappa)G(S_{KL-L})2^{2h(f)},$$

where $C(K,\kappa) = \frac{K-\kappa}{\kappa}K^{-\frac{K}{K-1}}$. $G(\Lambda)$ is the normalized second moment of a Voronoi cell of the lattice Λ and $G(S_{KL-L})$ is the normalized second moment of a sphere in $KL-L$ dimensions. We use our results to illustrate some relevant trade-offs that are made in configuring an MDC.

1. INTRODUCTION

We consider encoding of a memoryless source with probability density function f, differential entropy $h(f) <\infty$, and the mean squared-error distortion measure using K-channel multiple description coding (MDC) scheme. As a joint source-channel coding method, MDC aims at combating packet losses by exploiting network diversity. It generates a plurality of descriptions of a source sequence and transmits each description over independent erasure channel to the receiver side. The most common scenario is the case that all the channels are equivalent, which we will focus on in the paper. The system is designed in the way that the reconstruction quality gracefully improves when the number of received descriptions increases. This naturally prompts the question of how to distribute the redundant information among the descriptions efficiently to control the distortion v.s. number of received descriptions trade-off. Depending on the way of managing redundant information, existing MDC systems can be grouped into three categories: quantizer-based, transform-based and source-channel erasure codes based. Our work concentrates on lattice vector quantization which falls into the first category.

Since the pioneer work of [1] where a practical scalar quantization-based MDC (MDSQ) method is first proposed, many researches have been conducted focusing on designing efficient quantization-based MDC algorithms. The design of MDSQ is essentially converted to construct good index assignment matrices for two channel case (see [2], [3] and [1]) or index assignment arrangements for multi-channel case (see [4]). Especially, the designing problem of MDSQ for two channel case is well understood [5],[6]. Suppose a multiple-description encoder send information over each channel at a rate of R bits per sample. The performance of the system is measured by a three-tuple $(R, D^{(2,1)}, D^{(2,2)})$ where $D^{(2,1)}$ is one-channel distortion (or side distortion) and $D^{(2,2)}$ is two-channel distortion. The work of [6] showed that for an entropy-constrained multiple-description encoder, the distortions satisfy

$$\lim_{R\to\infty} D^{(2,2)} 2^{2R(1+a)} = \frac{1}{4}\left(\frac{2^{2h(f)}}{12}\right) \quad (1)$$

$$\lim_{R\to\infty} D^{(2,1)} 2^{2R(1-a)} = \left(\frac{2^{2h(f)}}{12}\right), \quad (2)$$

where $a \in (0,1)$ is the parameter which controls the redundant information between the two descriptions.

It is well known that vector quantization has the *space filling* advantage over scalar quantization [7]. This is because one has the freedom to construct cell shapes that are more "spherical" than a hypercube in higher dimensional space. Specifically, for the scenario of one channel entropy-coded at R bits per sample, when using an $L-$dimensional lattice Λ as a codebook the distortion $D_V(R)$ related with the distortion of $D_S(R)$ of scalar quantizer by

$$\lim_{R\to\infty} \frac{D_V(R)}{D_S(R)} = \frac{G(\Lambda)}{1/12},$$

where $G(\Lambda)$ is the normalized second moment of a Voronoi cell of the lattice Λ. It has been shown [8] that

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good lattices exist that satisfy $G(\Lambda) \rightarrow \frac{1}{2\pi e}$ as $L \rightarrow \infty$. Lattices are commonly used in the design of MDC algorithms (MDLVQ) to gain quantization efficiency over MDSQ [3]. As in MDSQ, the main design task in MDLVQ is to construct a good index assignment. The performance of a MDLVQ method for the two-channel case was analyzed in [3], culminating in the relation

$$\lim_{R\to\infty} D^{(2,2)} 2^{2R(1+a)} = \frac{1}{4} G(\Lambda) 2^{2h(f)}, \quad (3)$$

$$\lim_{R\to\infty} D^{(2,1)} 2^{2R(1-a)} = G(S_L) 2^{2h(f)}, \quad (4)$$

where $G(S_L)$ is the normalized second moment of a sphere in L dimensions. As compared to (1)–(2), both the central and side decoders exhibit a reduction in granular distortion. Surprisingly, the side distortion is characterized by $G(S_L)$ and is unrelated to the applied lattice structure.

An alternative characterization of the performance of a practical MDC system is the product of the central and side distortions. The theoretical lower bound of the product for a scalar Gaussian source with variance σ^2 satisfies [6]

$$D^{(2,2)} D^{(2,1)} \geq \frac{\sigma^4}{4} 2^{-4R}.$$

Suppose the rate R per channel is increased by $1/2$ bit, for a total increase of one bit. From (1)-(4) it is then seen that both the central and the side distortion decrease by $2^{-(1+a)}$ and $2^{-(1-a)}$, respectively. Although the decreases in rate differ, the product is $1/4$. It is well known that for a single-description high resolution quantizer, an extra bit reduces the distortion also by $1/4$. This naturally motivates the question of what happens to the product of the distortions for the K-channel MDC system.

A K-channel MDC scheme exploits additional network diversity to address packet loss, compared to the two-channel MDC scheme. The process of creating K descriptions increases the flexibility in designing the coding system. A general K-channel MDLVQ quantizer was first proposed in [9]. The method was designed for any lattice structure and any number of descriptions. The work was further extended in [10], where the search complexity for good index assignment is reduced significantly. [10] provides an asymptotic analysis but the geometrical interpretation of the result is not straightforward.

The goal of this paper is to provide a new performance analysis for a general method of constructing K-channel MDLVQ system, as described in [10]. The central distortion $D^{(K,K)}$ is easily evaluated within the framework of high-rate quantization theory. However, the analysis of side distortions is not trivial. Our new derivation is based on a geometrical argument, making it relatively straightforward to understand. It shows that the side distortions $D^{(K,\kappa)}$, $0 < \kappa < K$, are characterized by $G(S_{KL-L})$, the normalized second moment of a sphere in $KL - L$ dimensions.

2. PRELIMINARIES

Suppose a sequence of independent identically distributed (iid) random variables with probility density function (pdf) f are generated by an information source. We segment the data into L-dimensional vectors $X = (X_1, X_2, \ldots, X_L)^T$ and denote the pdf of the vector by f_X, which is thus given by

$$f_X = \Pi_{i=1}^{L} f(x_i).$$

The vector X is quantized to the nearest point λ_c in a central codebook $\Lambda_c \subset \mathbb{R}^L$. We denote the (central) quantization operation by $\lambda_c = \mathcal{Q}(X)$. Information about the central codeword λ_c is then embedded in K descriptions, which are transmitted independently across K erasure channels. This is performed through a labeling function α followed by entropy coding. The function α defines a bijective mapping from the cental codebook to K side codebooks, i.e. $\alpha : \Lambda_c \rightarrow \Lambda_0 \times \Lambda_1 \ldots \times \Lambda_{K-1}$. Let $\lambda|_0^{K-1}$ denote a K-tuple $(\lambda_0, \lambda_1, \ldots, \lambda_{K-1})$, where $\lambda_i \in \Lambda_i$. Thus, the labeling function links each element $\lambda_c \in \Lambda_c$ with a corresponding K-tuple $\lambda|_0^{K-1}$. We denote the i'th component of α as α_i, i.e. $\lambda_i = \alpha_i(\lambda_c)$, $i = 0, 1, \ldots, K-1$.

In this paper we analyze the behavior of MDC systems under the high-rate assumption. The assumption implies that the pdf of the source is approximately constant in a quantization cell. Gersho [11] conjectured that the optimal entropy-constrained high-rate vector quantizer for a uniform distribution over a convex bounded set has a partition whose quantization cells are congruent with some tessellating convex polytope. This establishes the basis for applying lattice geometry in vector quantization systems.

We study the scenario where both the central and side codebooks are lattices. The central codebook Λ_c is a lattice with a fundamental region of volume $\nu = \det(\Lambda_c)$. The K side codebooks are drawn from a geometrically-similar and clean sublattice $\Lambda_s \subseteq \Lambda_c$ [12] of index $N = |\Lambda_c/\Lambda_s|$, i.e., $\Lambda_i = \Lambda_s$, $i = 0, \ldots, K-1$. Thus, Λ_s has a fundamental region of volume νN. We consider the case that the channel conditions are symmetric, corresponding to an equal rate allocation across all K channels.

At the receiver side, if all the K descriptions are received, the inverse labeling function α^{-1} uniquely determines the central codeword $\mathcal{Q}(X)$. Because of packet loss, the decoder may receives only a subset of the K descriptions. Suppose κ of K descriptions are received. Let $\mathcal{L}^{(K,\kappa)}$ denote the set consisting of all the possible configurations. Each element $l \in \mathcal{L}^{(K,\kappa)}$ specifies a particular combination of the received descriptions, denoted as $\{\lambda_{l_j}, j = 1, 2, \ldots, \kappa\}$. There are $|\mathcal{L}^{(K,\kappa)}| = \binom{K}{\kappa}$ such combinations. In principle, there should be $\binom{K}{\kappa}$ decoding subsystems for a particular κ. To address the decoding complexity, a simple decoding rule was proposed in [9]. When $0 < \kappa < K$, the source X is reconstructed by averaging of the received descriptions, i.e.

$$\hat{x} = \frac{1}{\kappa} \sum_{j=1}^{\kappa} \lambda_{l_j}. \quad (5)$$

The distortion is measured as $\| x - \hat{x} \|^2$, where $\| \cdot \|$ denotes the l_2 norm. Note that this decoding process is in-consistent. If all K descriptions are received, the inverse mapping α^{-1} is used, but otherwise averaging is used. By allowing this decoding inconsistency, the design complexity for the mapping α (the index assignment) is significantly reduced. We use $D^{(K,\kappa)}$ to denote the (mean) distortion when κ out of K descriptions are received.

Generally speaking, once the side codebooks are fixed, the transmission rate per channel is determined [3]. On the other hand, fixing the central codebook specifies the central distortion $D^{(K,K)}$. However, the side distortions $D^{(K,\kappa)}$, $0 < \kappa < K$, depend on the labeling function α. The main task in designing practical MDLVQ systems is to find the α (the index assignment) that minimizes the side distortions.

3. INDEX ASSIGNMENT

In this section, we present a simple but efficient index-assignment method proposed in [10]. The simplicity of the method enables us to trace the geometrical properties of the index assignment.

The Voronoi cell of a lattice point $\lambda \in \Lambda$ is formally defined as

$$V(\lambda) = \left\{x : \| x - \lambda \|^2 \leq \| x - \lambda' \|^2, \lambda' \in \Lambda\right\}, \quad (6)$$

where the ties are broken in a predefined manner. We use subscripts to distinguish the Voronoi cells of different lattices, e.g. $V_c(\lambda_c)$ is the Voronoi cell for the central-lattice point $\lambda_c \in \Lambda_c$. To simplify the notation, a discrete Voronoi cell for every $\lambda_s \in \Lambda_s$ is defined as $V^d(\lambda_s) = V_s(\lambda_s) \cap \Lambda_c$. As Λ_s is a sublattice of Λ_c, the distribution of central points within every Voronoi cell of Λ_s is the same. This enables the index assignment to be designed by considering only the central points within one discrete Vononoi cell, e.g. $V^d(0)$. The index assignment can then be extended to cover all discrete Voronoi cells by translation,

$$\alpha(\lambda_c + \lambda_s) = \alpha(\lambda_c) + \lambda_s, \text{for all } \lambda_s \in \Lambda_s. \quad (7)$$

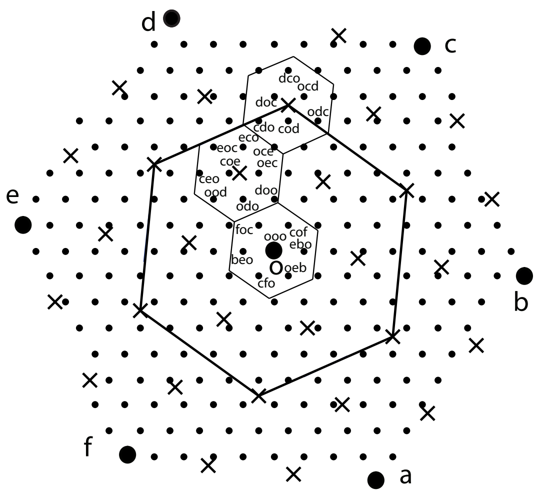

Figure 1. Three description index assignment for the lattice A_2 with index $N = 73$. Points of Λ_c, Λ_s and $\Lambda_{s/3}$ are denoted by $\cdot$, $\bullet$ and $\times$, respectively.

We start by introducing a so-called *scaled sublattice* $\Lambda_{s/K}$ [10], which is defined as

$$\Lambda_{s/K} = \frac{1}{K}\Lambda_s. \quad (8)$$

It is immediate that $\Lambda_s \subset \Lambda_{s/K}$. Denote Λ_s^K as the K-ary Cartesian product of Λ_s, i.e. $\Lambda_s^K = \prod_{i=0}^{K-1} \Lambda_s$. One can easily show that the centroid of any K-tuple from Λ_s^K is a scaled sublattice point. An onto mapping function β from Λ_s^K to $\Lambda_{s/K}$ can then defined as

$$\beta(\lambda|_0^{K-1}) = \frac{1}{K}\sum_{i=1}^{K} \lambda_i. \quad (9)$$

It computes the centroid of a K-tuple $\lambda|_0^{K-1} \in \Lambda_s^K$. It is obvious that each lattice point of $\Lambda_{s/K}$ is associated with many K-tuples. The scaled sublattice $\Lambda_{s/K}$ can be interpreted as a centroid distribution of the K-tuples over the space $\mathbb{R}^L$. Thus, by exploiting $\Lambda_{s/K}$, the β function provides a unified way to arrange the K-tuples used for index assignment. One observes that the K-tuples with the same centroid have different "spread", i.e. some are more compact than others. A distance criterion can be defined to measure the spread of a K-tuple [10]

$$J(\lambda|_0^{K-1}) = \sum_{i=0}^{K-1} \| \bar{\lambda}|_0^{K-1} - \lambda_i \|^2 , \quad (10)$$

where $\bar{\lambda}|_0^{K-1}$ denotes the centroid of this K-tuple. We refer to this criterion as *spread measurement*. Note that $\Lambda_{s/K}$ is obtained by scaling Λ_s by a factor K. This indicates that the fundamental Voronoi cell $V_s(0)$ contains K^L different scaled sublattice points up to translations $\{\lambda_s - \lambda'_s : \lambda_s, \lambda'_s \in \Lambda_s\}$. Depending on K, it might happen that the lattice points of $\Lambda_{s/K}$ lie on the cell boundary of Λ_s. The relationship between the central lattice and the scaled sublattice is studied in [10], which is characterized in the following proposition.

Proposition 3.1 *If Λ_s is a clean sublattice of Λ_c, then no central-lattice points of Λ_c lie on the cell boundary of $\Lambda_{s/K}$.*

The index assignment is performed with the aid of the scaled sublattice. First, the central point is quantized to the nearest point of $\Lambda_{s/K}$. This establishes the relationship that each point of $\Lambda_{s/K}$ is associated with many central points. *Proposition* 3.1 guarantees that no central points lie on the cell boundary of $\Lambda_{s/K}$, which simplifies the index assignment. To label the central points, the K-tuples with a centroid $\lambda_{s/K}$ can be ordered according to their cost by using (10). The central points within the Voronoi cell $V_{s/K}(\lambda_{s/K})$ are then assigned to the K-tuples from the ordered sequence of tuples. This is a natural choice as a K-tuple with small "spread" should be favored in the index assignment. In this case, when some descriptions are lost, the averaging operation in (5) likely results in a reconstruction point close to the corresponding central point. An example for a three-channel labeling scenario using a hexagonal lattice is displayed in Fig. 1. By applying (7), all central points can be labeled systematically. This approach guarantees that no K-tuple is reused, thus assuring that the function α is a one-to-one mapping.

4. ASYMPTOTICAL ANALYTIC PERFORMANCE

We now provide a new performance analysis for the presented index-assignment method. By algebraic manipulation, we associate K-tuples with points of a new lattice in $KL-L$ dimensions. This new geometric property then leads to a closed-form expression of the distortions.

First we derive expressions for the rate (in bits per dimension). Let R_c be the central rate required to address the central codebook Λ_c. Let $\mathcal{H}(\cdot)$ denote the entropy of a random variable. By exploiting high-rate quantization theory, $R_c = \mathcal{H}(\mathcal{Q}(X))/L$ can be approximated as

$$R_c \approx h(f) - (1/L)\log_2(\nu). \tag{11}$$

The transmission rate R per description of the multiple-description system can be evaluated by considering the quantity $\mathcal{H}(\alpha_i(Q(X)))/L$. Strictly speaking, the rate R is closely related with the index assignment approach. However, again by assuming high-rate quantization, it can be shown [3] that R has a simple expression, given as

$$R \approx h(f) - (1/L)\log_2(N\nu). \tag{12}$$

Note that the term $N\nu$ is simply the volume of the fundamental region $V_s(0)$ of the sublattice Λ_s. Thus, R is essentially determined by the side lattice codebook. From (11) and (12), it is seen that the relation between R_c and R is

$$R = R_c - (1/L)\log_2(N). \tag{13}$$

Since the total rate in the multiple-description system is $KR = KR_c - (K/L)\log_2(N)$, the rate overhead is given by $(K-1)R_c - (K/L)\log_2(N)$.

Next we study the central $D^{(K,K)}$ and side distortions $D^{(K,\kappa)}$, $0<\kappa<K$. The central distortion (per dimension) is determined by the central codebook Λ_c, which satisfies

$$D^{(K,K)} \approx G(\Lambda)\nu^{2/L}. \tag{14}$$

The regularity of the described labeling function leads to a simplicity of the expressions of the side distortions. We use $D_l^{(K,\kappa)}$ to denote the distortion for a particular configuration l from the set of configurations where κ out of K descriptions are received, i.e. $l \in \mathcal{L}^{(K,\kappa)}$. By applying (5), $D_l^{(K,\kappa)}$ can be expressed as

$$D_l^{(K,\kappa)} = \frac{1}{L}\sum_{\lambda_c\in\Lambda_c}\int_{V_c(\lambda_c)} f_X(x) \parallel x - \frac{1}{\kappa}\sum_{i=1}^{\kappa}\lambda_{l_i} \parallel^2 dx. \tag{15}$$

Thus, the (mean) side distortion $D^{(K,\kappa)}$ can be expressed in terms of $D_l^{(K,\kappa)}$ as

$$D^{(K,\kappa)} = \frac{1}{\binom{K}{\kappa}}\sum_{l\in\mathcal{L}^{(K,\kappa)}} D_l^{(K,\kappa)}. \tag{16}$$

Under the high rate assumption, one can show that the side distortion can be approximated as [9]

$$\begin{aligned} D^{(K,\kappa)} \quad &\approx \quad D^{(K,K)} + \frac{1}{\binom{K}{\kappa}}\frac{1}{NL}\sum_{\lambda_c\in V^d(0)}\sum_{l\in\mathcal{L}^{(K,\kappa)}} \\ &\Big[\parallel \lambda_c - \frac{1}{\kappa}\sum_{i=1}^{\kappa}\lambda_{l_i}\parallel^2\Big]. \end{aligned} \tag{17}$$

The second term in (17) can be simplified further (see [9], [10]), which leads to a new expression for the side distortion

$$D^{(K,\kappa)} \approx D^{(K,K)} + D_1 + \frac{K-\kappa}{K\kappa(K-1)}D_2 \tag{18}$$

where

$$D_1 = \frac{1}{NL}\sum_{\lambda_c\in V^d(0)} \parallel \lambda_c - \bar{\lambda}(\lambda_c)\parallel^2, \tag{19}$$

$$D_2 = \frac{1}{NL}\sum_{\lambda_c\in V^d(0)}\sum_{i=0}^{K-1} \parallel \lambda_i(\lambda_c) - \bar{\lambda}(\lambda_c)\parallel^2, \tag{20}$$

where the K-tuple assigned to λ_c is $\{\lambda_i(\lambda_c)\}_{i=0}^{K-1}$ and $\bar{\lambda}(\lambda_c) = \frac{1}{K}\sum_{j=0}^{K-1}\lambda_j(\lambda_c)$. Interestingly, the side distortion is not dependent on the source pdf. It is fully determined by the lattice structure and the index assignment.

Next, we investigate the index assignment to derive analytic expressions for the side distortions. From (10) it follows that D_2 is essentially the summation of the spread measurements. As the exploited K-tuples are searched and arranged w.r.t. their centroids described by $\Lambda_{s/K}$, D_2 can be further decomposed with the aid of $\Lambda_{s/K}$. We consider the spread measurement for a general K-tuple. Suppose the generator matrix of the sublattice Λ_s is γG, where G is selected such that the matrix $M = GG^T$ is an integer matrix. A lattice generated with such matrix is called an *integral lattice* [12]. Thus, we have $N\nu = \sqrt{\gamma^{2L}|M|}$, where $|M|$ denotes the determinant of M [12]. Denote the K side lattice codebooks as

$$\Lambda_i = \left\{Z_i^T\gamma G| \quad \forall Z_i\in\mathbb{Z}^L\right\},\ i=0,1,\ldots,K-1, \tag{21}$$

where Z_i specifies the coordinates in the i-th codebook. From (8), the scaled sublattice takes the form of

$$\Lambda_{s/K} = \left\{\frac{1}{K}W^T\gamma G| \quad \forall W\in\mathbb{Z}^L\right\}. \tag{22}$$

Using (21) and (22), the spread measurement in (10) can then be reformulated as

$$J(\lambda|_0^{K-1}) = \sum_{i=0}^{K-1} \parallel Z_i^T\gamma G - \frac{1}{K}W^T\gamma G\parallel^2 \tag{23}$$

$$\text{subject to } \frac{1}{K}W^T\gamma G = \frac{1}{K}\sum_{i=0}^{K-1} Z_i^T\gamma G.$$

Using algebra, it can be shown that (23) can be further simplified to

$$J(\lambda|_0^{K-1}) = \gamma^2(Z-\check{W})^T M^*(Z-\check{W}), \tag{24}$$

where

$$M^* = \begin{bmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \ldots & 1 & 2 \end{bmatrix} \otimes M, \tag{25}$$

$$Z = [\ Z_0^T \quad Z_1^T \quad \ldots \quad Z_{K-2}^T\ ]^T, \tag{26}$$

$$\check{W} = \frac{1}{K}[\ W^T \quad W^T \quad \ldots \quad W^T\ ]^T. \tag{27}$$

The operator $\otimes$ is the Kronecker product [13]. The matrix before $\otimes$ is the Gram matrix of an A_{K-1} lattice [12]. The matrix M^* is a symmetric matrix with dimensionality $(K-1)L$. Equation (24) essentially defines a new lattice Λ_{tuple} with a translation $s = \gamma G^{*T}\check{W}$, where $M^* = G^*G^{*T}$. We refer the new lattice as a *tuple* lattice since K-tuples are essentially associated with lattice points. The Gram matrix of Λ_{tuple} is $\gamma^2 M^*$. When $L = 1$, the tuple matrix Λ_{tuple} reduces to A_{K-1} lattice. The expression in (24) can be interpreted as the squared l_2 norm of a point of $\Lambda_{tuple} - s$. The search for good K-tuples is essentially reduced to selecting the points of $\Lambda_{tuple} - s$ with small squared l_2 norms. The translation vector s is fully determined by a scaled sublattice point. By observing (24) and (27), one can see that when a component of the coordinate vector W is modified by adding a multiple of K, the translated lattice geometry $\Lambda_{tuple} - s$ remains the same. There are K^L different translated lattice geometries. Informally speaking, this is because the relative arrangement of the scaled sublattice points and Λ_s exhibits periodicity over space.

Upon relating K-tuples with points of a tuple lattice Λ_{tuple}, we are ready to analyze the side distortions (18). As the points from both Λ_c and $\Lambda_{s/K}$ are uniformly distributed over the space, we can assume that each Voronoi cell $V_{s/K}(\lambda_{s/K})$ contains approximately N/K^L central lattice points. This holds when the index value N is large. Denote the N/K^L K-tuples exploited to label the central points within a Voronoi cell $V_{s/K}(\lambda_{s/K})$ as $\lambda|_0^K(i)$, $i = 0, 1, \ldots, N/K^L - 1$, where $\bar{\lambda}|_0^K(i) = \lambda_{s/K}$. The sum of spread measurements of these N/K^L K-tuples can be parameterized by studying Λ_{tuple}. As N is sufficiently large, the N/K^L selected points of $\Lambda_{tuple} - s$ can be approximated by the N/K^L selected points of Λ_{tuple} in computing the sum of spread measurements. The analysis can be simplified by only considering the situation that $\bar{\lambda}|_0^{K-1}(i) = 0$, $i = 0, 1, \ldots, N/K^L - 1$. Denote the theta series [12] of Λ_{tuple} as $\Theta_{\Lambda_{tuple}}(z) = \gamma^2 \sum_{j=0}^{\infty} B_j q^j$, where $q = e^{iz}$. The coefficient B_j indicates the number of points in the j-th shell. Thus, we can write

$$\sum_{i=0}^{N/K^L-1} J(\lambda|_0^K(i)) = \gamma^2 \sum_{j=0}^{E} jB_j, \tag{28}$$

where we assume that $N/K^L = \sum_{j=0}^{E} B_j$. The parameter E indicates the maximum shell index. To further simplify (28), we first introduce a widely exploited approximation [3] in the following proposition.

Proposition 4.1 *Let ν' be the volume of a Voronoi cell of an integral lattice Λ in $\mathbb{R}^L$. Denote $\mathcal{V}_L$ as the volume of a sphere of unit radius in $\mathbb{R}^L$. The number of lattice points $S(n)$ in the first n shells of the lattice Λ is approximately $S(n) = \frac{\mathcal{V}_L n^{\frac{L}{2}}}{\nu'}(1 + o(1))$, where* $\lim_{n\to\infty} o(1) = 0$.

The result of *Proposition* 4.1 provides a simple expression for the number of points within the first n shells of a lattice. This enables one to approximate the two terms $\sum_{j=0}^{E} B_j$ and $\sum_{j=0}^{E} jB_j$ using simple expressions [3]. After some algebra, one can show that (28) can be approximated as

$$\begin{aligned} \sum_{i=0}^{N/K^L-1} J(\lambda|_0^K(i)) \quad &\approx \quad \gamma^2 \frac{|M^*|^{\frac{1}{(K-1)L}}}{\mathcal{V}_{(K-1)L}^{\frac{2}{(K-1)L}}} \cdot \frac{(K-1)L}{(K-1)L+2} \\ &\cdot K^{-L-\frac{2}{K-1}} \cdot N^{1+\frac{2}{(K-1)L}}, \end{aligned} \tag{29}$$

where $|M^*| = K^L(\frac{N\nu}{\gamma^L})^{2(K-1)}$. It is known that $\mathcal{V}_{(K-1)L}$ can be expressed in terms of $G(S_{KL-L})$, the normalized second moment of a sphere in $(K-1)L$ dimensions, by

$$\mathcal{V}_{(K-1)L}^{\frac{2}{(K-1)L}} = \frac{1}{G(S_{KL-L})((K-1)L+2)}. \tag{30}$$

Based on (29) and (30), the distortion D_2 takes the form

$$\begin{aligned} D_2 \quad \approx \quad & (K-1)K^{-\frac{1}{K-1}}(N\nu)^{\frac{2}{L}} G(S_{KL-L}) \\ & N^{\frac{2}{(K-1)L}}(1 + o(1)). \end{aligned} \tag{31}$$

From (19) and (20), it is seen that D_1 and D_2 are similar. Thus, a similar analysis can be performed on quantity D_1, resulting in

$$D_1 \approx K^{-2}(N\nu)^{\frac{2}{L}} G(S_L)(1 + o(1)). \tag{32}$$

The side distortion $D^{(K,\kappa)}$ is thus fully specified.

We now study the relationship between rates and distortions. First, the central distortion can be expressed in terms of R_c as

$$D^{(K,K)} = G(\Lambda)2^{2(h(f)-R_c)}.$$

Let $N = 2^{La(K-1)R}$. It follows from (13) that $R_c = R[1 + a(K-1)]$. Using (11) and (14) the central distortion can be rewritten as

$$D^{(K,K)} = G(\Lambda)2^{2h(f)}2^{-2R[1+a(K-1)]}. \tag{33}$$

Using (11), (31) and (32), the two distortions D_1 and D_2 can be expressed in terms of R as

$$\begin{aligned} &\lim_{R\to\infty} D_1 2^{2R} = K^{-2} G(S_L) 2^{2h(f)} \\ &\lim_{R\to\infty} D_2 2^{2R(1-a)} = (K-1)K^{-\frac{1}{K-1}} G(S_{KL-L}) 2^{2h(f)}. \end{aligned}$$

Note that the distortion D_2 decays slower than D_1 and $D^{(K,K)}$, and thus dominates the side distortion $D^{(K,\kappa)}$. The final result of $D^{(K,\kappa)}$ is

$$\lim_{R\to\infty} D^{(K,\kappa)} 2^{2R(1-a)} = \frac{K-\kappa}{\kappa} K^{-\frac{K}{K-1}} G(S_{KL-L}) 2^{2h(f)}. \tag{34}$$

The parameter a controls the redundant information between the K descriptions. We can see that the side distortions $D^{(K,\kappa)}$, $1 \leq \kappa < K$ are characterized by the normalized second moment of a sphere in $KL-L$ dimensions. Suppose each R is increased by $1/K$ bit, then the central distortion is decreased by $2^{-\frac{2}{K}[1+a(K-1)]}$ and the side distortion $D^{(K,\kappa)}$ by $2^{-\frac{2}{K}(1-a)}$ for any $a \in (0,1)$. Thus, the product of the decreasing factors is

$$2^{-\frac{2}{K}[1+a(K-1)]}2^{-\frac{2}{K}(1-a)(K-1)} = \frac{1}{4}. \qquad (35)$$

This shows that K-channel multiple-description quantizer is efficient as compared to (1)–(4) for the two channel case. The product of the distortions takes the form of

$$\begin{aligned}&\lim_{R\to\infty}\prod_{i=1}^{K} D^{(K,i)}2^{2RK}\\ &= K^{-K}G(\Lambda)[G(S_{KL-L})]^{K-1}2^{2Kh(f)}. \qquad (36)\end{aligned}$$

It is seen that, as expected, the parameter a is not involved in the product, which is consistent with the two-channel results in [3].

One observes that when κ increases from 1 to $K-1$, the side distortion is reduced by a factor $\frac{K-\kappa}{\kappa}$. The distortion reduction rate from $K-1$ to K exhibits a singularity. This might be due to the fact that a decoding inconsistency is introduced in the system. It is known that as $L \to \infty$, $G(\Lambda) \to 1/2\pi e$ and $G(S_L) \to 1/2\pi e$, which render minimum central and side distortions. This allows us to study the distortion loss due to dimensionality. From (33), the loss for $D^{(K,K)}$ can be expressed as

$$\lim_{R\to\infty} D^{(K,K)}(R,L)/D^{(K,K)}(R,\infty) = 2\pi e G(\Lambda).$$

This indicates that the loss in central distortion is lattice dependent, but channel-number independent. By considering (34), the side distortion loss takes the form of

$$\lim_{R\to\infty} D^{(K,\kappa)}(R,L)/D^{(K,\kappa)}(R,\infty) = 2\pi e G(S_{KL-L}),$$

where $1 \le \kappa < K$. It is seen that the side distortion loss is independent of the number of received descriptions. The loss depends on the dimension of the lattice and the channel number K. Again, a singularity appears from $\kappa = K-1$ to K. Fig. 2 displays the distortion loss vs dimensionality for different number of channels.

5. CONCLUSION

We have studied a general MDLVQ scheme from a geometrical view point. We have found that K-tuples used for index assignment can be associated with points of another lattice of higher dimensionality than that of the quantization space. With the aid the newly obtained geometry, we showed that the side distortions are characterized by the normalized second moment of a sphere in the same dimensionality as that of the new lattice. A major result is that the product of all distortions of the MDLVQ system is asymptotically independent of the redundancy between the descriptions.

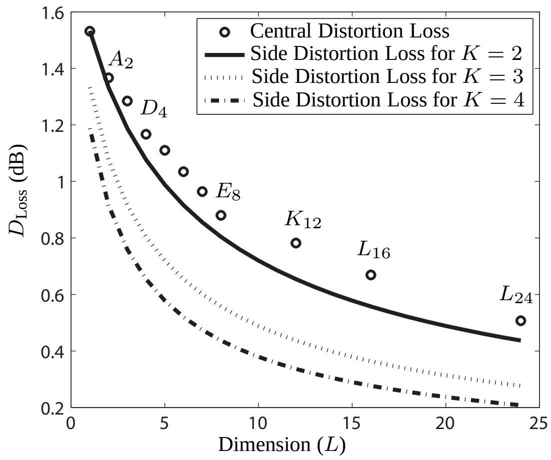

Figure 2. The central and side distortion loss as a function of the lattice dimensionality L.

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