On the EXIT Characteristics of Feed Forward Convolutional Codes

Laurent Schmalen, Peter Vary
Institute of Communication Systems
and Data Processing (ind)
RWTH Aachen University, Germany
{schmalen,vary}@ind.rwth-aachen.de

Marc Adrat
FGAN e.V.
FKIE/KOM
Wachtberg, Germany
adrat@fgan.de

Thorsten Clevorn
Infineon Technologies
COM Development Center NRW
Duisburg, Germany
thorsten.clevorn@infineon.com

***Abstract*— In our contribution we analyze the EXIT characteristics of feed forward convolutional codes employed in a parallel concatenated Turbo scheme or as inner component in a serially concatenated scheme. It turns out that the EXIT characteristics reveal some general limitations. First we give an illustrative reason for the limited capabilities of feed forward and non-terminated recursive convolutional codes and secondly, we provide novel analytical means to determine the maximum mutual information available at the extrinsic output of feed forward convolutional decoders. This upper bound for the mutual information is only dependent on the channel quality and on the Hamming weight of the impulse response of the convolutional code. The paper concludes by determining the maximum mutual information for a wide range of channel conditions and code parameters, showing for which codes and channel conditions a maximum mutual information of ≈ 1 bit can be reached.**

I. Introduction

With the discovery of Turbo codes [1], [2], [3] channel coding close to the Shannon limit becomes possible with moderate computational complexity. The convergence of Turbo processes can be analyzed using EXIT charts [4], [5]. EXIT charts visualize the exchange of *extrinsic information* between the constituent decoders in a concatenated system. Each decoder is represented by an EXIT characteristic, which describes the *extrinsic information transfer* (EXIT) from the *a priori* input to the *extrinsic* output of the decoder. In the original papers [4], [5], S. ten Brink proposed to determine EXIT characteristics by histogram measurements. The *a priori* input is modelled by a Gaussian process with a specific mean and variance, and the resulting *extrinsic* output values are recorded in histograms. From the histograms for different Gaussian inputs it is straightforward to compute the EXIT characteristics [5]. The original EXIT chart papers mainly considered recursive convolutional codes.

On the other hand, feed forward convolutional codes have been used in a variety of existing communication systems, e.g., GSM and UMTS. However, due to their suboptimal performance when used in iterative decoders [3], not much analysis has been performed for Turbo-like systems employing feed forward convolutional codes as inner channel codes. However, it can be beneficial to perform iterative decoding also in existing systems. For instance, the application of iterative source-channel decoding might be advantageous in speech transmission systems like GSM and UMTS [6], [7], [8]. To analyze such systems, EXIT charts can be a powerful tool.

In this paper we analyze the EXIT charts of feed forward convolutional codes, especially the property that no perfect extrinsic information can be generated even if perfect *a priori* knowledge is available. The mutual information of the extrinsic output is upper bounded, depending on the code and the channel quality. First, we give an easy explanation of this behavior using the Trellis diagram of the convolutional code and secondly, we show by analytical means that the maximum attainable mutual information only depends on the channel quality and the Hamming weight of the impulse response of the code. We also give an expression to calculate this mutual information.

The paper is organized as follows. Section II introduces the EXIT characteristics of feed forward convolutional codes and shows in a simulation example the property that the mutual information is upper bounded. Section III-A gives an explanation on this behavior and in the Section III-B, an analytical expression for the maximum attainable mutual information is derived. Finally, Section IV shows that this analytical expression matches the observations of Section II.

II. EXIT Charts of Feed Forward Convolutional Codes

Introduced in [4], EXIT charts have become an important tool for the convergence analysis of concatenated systems with iterative evaluation of extrinsic information at the receiver. EXIT characteristics plot the mutual information I_E of the extrinsic output of a decoder as a function of the mutual information I_A of the *a priori* input.

To clarify notations, Fig. 1 shows the block diagram of an EXIT chart measurement circuit [9] for a feed forward convolutional code employed in a parallel concatenated (PC) iterative decoding scheme or as inner component in a serially concatenated (SC) iterative decoding scheme. A binary source S generates a vector $\mathbf{u}$ of binary data bits. The vector $\mathbf{u}$ is encoded to a vector $\mathbf{x}$ using a convolutional encoder. After transmission over a communication channel the MAP SISO decoder [10] receives a possibly noisy vector $\mathbf{y}$. Here, the communication channel consists of BPSK modulation with symbol energy $E_{\mathrm{s}} = 1$, AWGN with noise

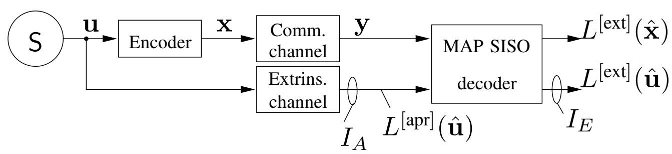

Fig. 1. Block diagram showing the measurement of the EXIT chart of a MAP SISO convolutional decoder [9]

variance $\sigma_n^2 = N_0/2$, BPSK demodulation, and conversion to L-values [2]. The MAP SISO decoder receives additional *a priori* information on the data bits $\mathbf{u}$ from the extrinsic output of the (notional) second constituent decoder of the iterative decoding scheme. This information is modelled by the *extrinsic channel* which adds Gaussian noise with defined mean and variance [5], [9] to the bipolar representation of those bits. Note that the inputs and the outputs of the MAP SISO decoder are represented as L-values. The MAP SISO decoder outputs extrinsic information on the data bits $L^{[\text{ext}]}(\hat{\mathbf{u}})$ and extrinsic information on the encoded bits $L^{[\text{ext}]}(\hat{\mathbf{x}})$. In our case, the mutual information of interest to compute the EXIT charts is the *a priori* mutual information $I_A = I(U; L_{\hat{U}}^{[\text{apr}]})$ and the extrinsic mutual information $I_E = I(U; L_{\hat{U}}^{[\text{ext}]})$. Note that $\mathbf{u}$ is a realization of the random process U and accordingly, $L^{[\text{ext}]}(\hat{\mathbf{u}})$ and $L^{[\text{apr}]}(\hat{\mathbf{u}})$ are realizations of the corresponding random processes $L_{\hat{U}}^{[\text{ext}]}$ and $L_{\hat{U}}^{[\text{apr}]}$.

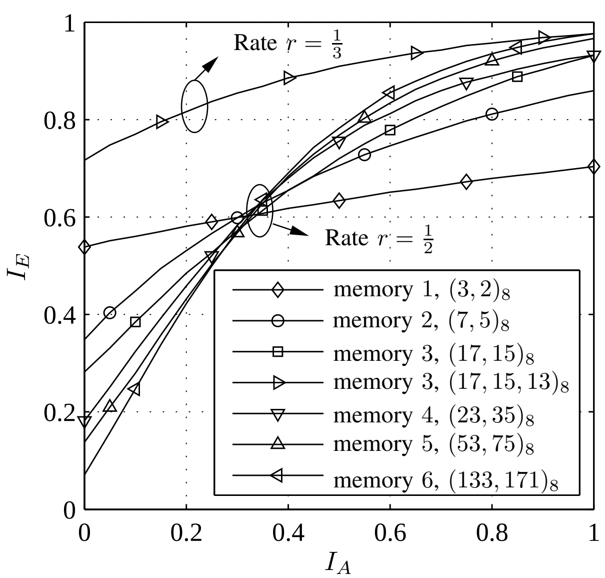

Fig. 2. EXIT characteristics of different feed forward convolutional codes for $E_s/N_0 = 1/(2\sigma_n^2) = -5\,\text{dB}$

Figure 2 shows the EXIT characteristics of different rate $1/2$ and rate $1/3$ feed forward convolutional codes acting in a parallel concatenated system or as inner code in a serially concatenated system, i.e., the input to the decoder consists of the *a priori* knowledge on the data bits and of the received channel values of the encoded bits; the decoder generates extrinsic information $L^{[\text{ext}]}(\hat{\mathbf{u}})$ for the data bits $\mathbf{u}$. As visible in Fig. 2, the characteristics do not reach the point $(I_A = 1\,\text{bit}\ , I_E = 1\,\text{bit})$ which is needed for perfect decoding. Table I lists the maximum mutual information for perfect *a priori* knowledge (i.e., $I_A = 1\,\text{bit}$) for the codes of Fig. 2. An easy to understand explanation of this behavior and an expression to analytically compute the mutual information I_E for $I_A = 1\,\text{bit}$ is given in Section III.

TABLE I
MEASURED MAXIMUM MUTUAL INFORMATION FOR THE CODES IN FIG. 2 AND $E_s/N_0 = 1/(2\sigma_n^2) = -5\,\text{dB}$

Generator polynomials	$I_E\big\rvert_{I_A=1}$
$(3,2)_8$	0.7038
$(7,5)_8$	0.8595
$(17,15)_8$	0.9316
$(17,15,13)_8$	0.9764
$(23,35)_8$	0.9324
$(53,75)_8$	0.9665
$(133,171)_8$	0.9762

III. Maximum Attainable Mutual Information

It has already been observed in [11] that the EXIT characteristics of feed forward convolutional codes do not reach $I_E = 1\,\text{bit}$ for $I_A = 1\,\text{bit}$. The explanation given is based on the fact that the coupling of the bits is limited by the constraint length of the code. We give a more detailed explanation of this behavior using the Trellis representation of the convolutional code. In Section III-B, we consider the problem theoretically and provide an analytical solution for the maximum attainable mutual information.

A. Illustrative Explanation

The behavior of imperfect mutual information if perfect *a priori* knowledge is available can best be visualized using the Trellis representation of the convolutional code. Figure 3 depicts parts of the Trellis diagram of a memory $J = 2$ feed forward convolutional code. Without loss of generality (due of the linearity of the code), it can be assumed that the all-zero path has been encoded and transmitted. Determining the extrinsic information for $\hat{u}_k$ at time instant k means determining the probability that the estimated data bit $\hat{u}_k$ at time instant k is either 0 or 1 if *a priori* information on the data bits u_k is available at all time instants except for k. If we take a look at Fig. 3, we see immediately that, if it is perfectly known that the all-zero data sequence has been sent ($I_A = 1\,\text{bit}$), the decision at instant k cannot be determined by purely considering the *a priori* knowledge at all time instants except k. However, in order to compute the extrinsic output at

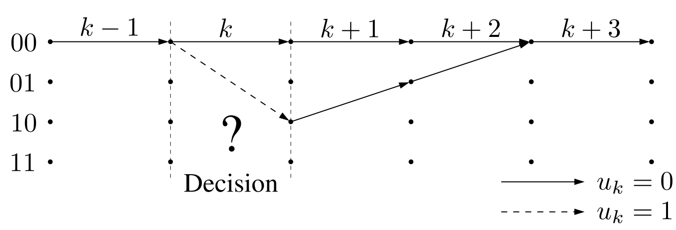

Fig. 3. Extrinsic Decision if perfect *a priori* knowledge is available in the case of a memory $J = 2$ feed forward convolutional code

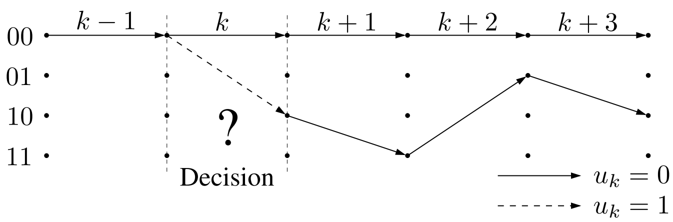

Fig. 4. Extrinsic Decision if perfect *a priori* knowledge is available in the case of a memory $J = 2$ recursive convolutional code

instant k, the decision on $\hat{u}$ has to be made using the channel output only. This decision is not influenced by the *a priori* knowledge as the (perfectly known) inputs at instants $k+1$, $k+2$, etc. lead to the same inner state of the convolutional encoder after J inputs (due to the non-recursive structure of the code).

In the case of recursive convolutional codes, however, a different data input u_k at time instant k and identical, perfectly known inputs at subsequent instants do not lead to the same state after J inputs, resulting from the recursiveness of the encoder as shown in Fig. 4. A different decision at instant k, followed by perfectly known data bits leads to a different Trellis path which does not end in the same state. Thus, if the encoder is terminated (i.e., the encoding stops in a defined state) and the input vector $\mathbf{u}$ is of finite length, perfect *a priori* knowledge leads to a non-ambiguous decision on the extrinsic output at instant k. If the recursive code is not terminated, similar effects as in the case of feed forward codes are observed, i.e., no perfect extrinsic information can be generated even if perfect *a priori* knowledge is available. Nevertheless, the remainder of this paper will only focus on feed forward convolutional codes.

B. Theoretical Results

Definition 3.1: Let C be a rate $\mathcal{R} = 1/N$ feed forward convolutional encoder with time-domain generator matrix $\mathbf{G}$. Let $\mathbf{u}^{(1)} = (1, 0, \ldots, 0)$ be a weight one input vector of length $J+1$. The vector $\mathbf{u}^{(1)}$ is encoded by C to the code vector $\mathbf{x}^{(1)} = \mathbf{u}^{(1)}\mathbf{G} = \left(x_1^{(1)}, x_2^{(1)}, \ldots\right)$ with $x_k^{(1)} \in \{0; 1\}$. The vector $\mathbf{x}^{(1)}$ is also denoted as the impulse response of the convolutional code. The Hamming weight of the impulse response vector $\mathbf{x}^{(1)}$ is defined as $\bar{D}_{\mathsf{C}}$ and it holds

$$\bar{D}_{\mathsf{C}} = \sum_{k=0}^{N(J+1)} x_k^{(1)} . \qquad (1)$$

Theorem 3.2: Given a rate $\mathcal{R} = 1/N$ feed forward convolutional code C with memory J and transmission over an AWGN channel with noise variance σ_n^2, the L-values of the extrinsic MAP SISO decoder output $L^{[\text{ext}]}(\hat{\mathbf{u}})$ show a Gaussian distribution with mean $\mu_E = 2\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$ and variance $\sigma_E^2 = 2\mu_E = 4\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$ if perfect *a priori* knowledge on the equiprobable data bits $\mathbf{u}$ is available (i.e., $I_A = 1\,\text{bit}$).

Proof: (*This proof uses intermediate results from [12]*) Let $\mathbf{u}^{(0)} = (u_1^{(0)}, \ldots, u_{J+1}^{(0)}) = (0, \ldots, 0)$ be the all zero vector and $\mathbf{u}^{(1)} = (u_1^{(1)}, u_2^{(1)} \ldots, u_{J+1}^{(1)}) = (1, 0, \ldots, 0)$ a weight one input vector. The length of the input vectors can be restricted to $J+1$, as after J identical inputs, the feed forward convolutional encoder will have the same inner state. Let $\mathbf{G}$ be the time domain generator matrix of the feed forward convolutional code C. Encoding both vectors $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ with C produces the outputs

$$\mathbf{x}^{(0)} = \mathbf{u}^{(0)}\mathbf{G} = \left(x_1^{(0)}, \ldots, x_{N(J+1)}^{(0)}\right) = (0, 0, \ldots, 0)$$
$$\mathbf{x}^{(1)} = \mathbf{u}^{(1)}\mathbf{G} = \left(x_1^{(1)}, \ldots, x_{N(J+1)}^{(1)}\right) .$$

Let $\mathbf{u}_{\setminus 1}^{(i)} = \left(u_2^{(i)}, \ldots, u_{J+1}^{(i)}\right) = (0, 0, \ldots, 0),\ i \in \{0; 1\}$, denote the vector of length J which does not contain the first element of either $\mathbf{u}^{(0)}$ or $\mathbf{u}^{(1)}$. Due to the linearity of the convolutional code, it is sufficient to perform the proof for the all-zero vector only.

The encoded vector $\mathbf{x}^{(0)}$ of length $N(J+1)$ is BPSK modulated onto the vector $\ddot{\mathbf{x}}^{(0)}$ with elements $\ddot{x}_k^{(0)} = 1 - 2x_k^{(0)} = +1,\ k = 1, 2, \ldots, N(J+1)$ and $\mathbf{x}^{(1)}$ is modulated onto $\ddot{\mathbf{x}}^{(1)}$. After transmission over a channel with additive white Gaussian noise of zero mean and variance $\sigma_n^2 = N_0/2$, the vector $\ddot{\mathbf{y}}^{(0)}$ is received with $\ddot{\mathbf{y}}^{(0)} = \ddot{\mathbf{x}}^{(0)} + \mathbf{n}$, and $\mathbf{n} = \left(n_1, \ldots, n_{N(J+1)}\right)$ denoting the noise vector. The extrinsic outputs of the MAP SISO decoder are the probabilities that the decoded data bit $\hat{u}_k$ is either 0 or 1 under the condition that *a priori* knowledge on all data bits except the one at position k and the entire received sequence $\ddot{\mathbf{y}}^{(0)}$ are available. Without loss of generality, it is sufficient to consider only the first bit position of the vectors, as a consequence of the linearity of convolutional codes. Using the Bayes theorem and the assumption that the data bits u_k are equiprobable, the extrinsic probabilities can be expressed as (with $\ell \in \{0; 1\}$) [12]

$$P(\hat{u}_1 = \ell | \ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\setminus 1}^{(0)}) = \frac{p\left(\ddot{\mathbf{y}}^{(0)} | \mathbf{u}^{(\ell)}\right)}{p\left(\ddot{\mathbf{y}}^{(0)} | \mathbf{u}^{(0)}\right) + p\left(\ddot{\mathbf{y}}^{(0)} | \mathbf{u}^{(1)}\right)}$$

with

$$p\left(\ddot{\mathbf{y}}^{(0)} | \mathbf{u}^{(0)}\right) = \left(\frac{1}{\sqrt{2\pi}\sigma_n}\right)^{N(J+1)} \cdot \prod_{\kappa=1}^{N(J+1)} \mathrm{e}^{-\frac{n_\kappa^2}{2\sigma_n^2}}$$
$$p\left(\ddot{\mathbf{y}}^{(0)} | \mathbf{u}^{(1)}\right) = \left(\frac{1}{\sqrt{2\pi}\sigma_n}\right)^{N(J+1)} \cdot \prod_{\kappa=1}^{N(J+1)} \mathrm{e}^{-\frac{(n_\kappa + d_\kappa)^2}{2\sigma_n^2}}$$

and d_k elements of the vector $\mathbf{d} = \ddot{\mathbf{x}}^{(0)} - \ddot{\mathbf{x}}^{(1)}$, i.e., $d_k \in \{0; +2\}$. Using vector notation, the extrinsic probabilities can be expressed as

$$P(\hat{u}_1 = 0 | \ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\setminus 1}^{(0)}) = \frac{\exp\left(\frac{|\mathbf{n}+\mathbf{d}|^2 - |\mathbf{n}|^2}{2\sigma_n^2}\right)}{1 + \exp\left(\frac{|\mathbf{n}+\mathbf{d}|^2 - |\mathbf{n}|^2}{2\sigma_n^2}\right)} \qquad (2)$$

$$P(\hat{u}_1 = 1 | \ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\setminus 1}^{(0)}) = \frac{1}{1 + \exp\left(\frac{|\mathbf{n}+\mathbf{d}|^2 - |\mathbf{n}|^2}{2\sigma_n^2}\right)} \qquad (3)$$

with $|\mathbf{n}|^2 = \sum_{\kappa=1}^{N(J+1)} n_k^2$ and $|\mathbf{n}+\mathbf{d}|^2$ defined similarly. By using (2) and (3) the extrinsic L-values $L^{[\text{ext}]}(\hat{\mathbf{u}})$ can be determined as

$$\begin{aligned} L^{[\text{ext}]}(\hat{u}_1) &= L(\hat{u}_1|\ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\backslash 1}^{(0)}) \qquad (4)\\ &= \ln\left(\frac{P(\hat{u}_1 = 0|\ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\backslash 1}^{(0)})}{P(\hat{u}_1 = 1|\ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\backslash 1}^{(0)})}\right)\\ &= \frac{|\mathbf{n}+\mathbf{d}|^2 - |\mathbf{n}|^2}{2\sigma_n^2}\,. \end{aligned}$$

The factor $|\mathbf{n}+\mathbf{d}|^2 - |\mathbf{n}|^2$ can be further simplified

$$\begin{aligned} |\mathbf{n}+\mathbf{d}|^2 - |\mathbf{n}|^2 &= \sum_{\kappa=1}^{N(J+1)} (n_\kappa + d_\kappa)^2 - \sum_{\kappa=1}^{N(J+1)} n_\kappa^2\\ &= 2\sum_{\kappa=1}^{N(J+1)} n_\kappa d_\kappa + \sum_{\kappa=1}^{N(J+1)} d_\kappa^2\\ &= 2\sum_{\kappa=1}^{N(J+1)} n_\kappa d_\kappa + 4\bar{D}_{\mathsf{C}}\,, \end{aligned}$$

using the fact that $\sum_{\kappa=0}^{N(J+1)} d_\kappa^2 = 4\bar{D}_{\mathsf{C}}$ (see definition of $\bar{D}_{\mathsf{C}}$). Therefore, we obtain

$$L(\hat{u}_1|\ddot{\mathbf{y}}^{(0)}, \mathbf{u}_{\backslash 1}^{(0)}) = 2\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2} + \sum_{\kappa=1}^{N(J+1)} \frac{d_\kappa}{\sigma_n^2} n_\kappa\,. \qquad (5)$$

Equation 5 states that the L-values of the extrinsic output are composed by the sum of weighted Gaussian distributed noise values and an offset $2\bar{D}_{\mathsf{C}}/\sigma_n^2$. A random process composed by sum of Gaussian distributed processes is again a Gaussian process [13]. The mean of the resulting process is the sum of the means of the sub-processes and the resulting variance is the sum of the sub-processes' variances. As the noise samples n_k have zero mean, the process resulting from the sum of all noise samples has zero mean. As a consequence, the mean of the L-values is only determined by the offset in (5) and the mean μ_E of the L-value distribution is thus

$$\mu_E = 2\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}\,.$$

The variance of the noise samples n_κ is σ_n^2, but as the noise samples n_κ are scaled by d_κ/σ_n^2, the variances of the scaled samples in the sum (5) amount to $\sigma_n^2 \cdot \frac{d_\kappa^2}{\sigma_n^4} = d_\kappa^2/\sigma_n^2$. Therefore, the total variance σ_E^2 of the L-value distribution is

$$\sigma_E^2 = \sum_{\kappa=1}^{N(J+1)} \frac{d_\kappa^2}{\sigma_n^2} = \frac{1}{\sigma_n^2}\sum_{\kappa=1}^{N(J+1)} d_\kappa^2 = 4\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2} = \frac{\mu_E}{2}\,.$$

■

Corollary 3.3: Given a feed forward convolutional code C, the hard decision bit error probability of the extrinsic output after transmission over an AWGN channel with noise variance σ_n^2 and after MAP SISO decoding is given by $P_b^{[\text{ext}]}\big|_{I_A=1} = \frac{1}{2}\,\text{erfc}\left(\frac{\sqrt{\bar{D}_{\mathsf{C}}}}{\sqrt{2}\sigma_n}\right)$ under the condition that perfect *a priori* knowledge on the data bits $\mathbf{u}$ is available ($I_A = 1$ bit).

Proof: Due to the linearity of the convolutional code C, it can be assumed, without loss of generality, that the all-zero codeword has been sent. It is known from Theorem 3.2 that the pdf of the extrinsic information, given that the all-zero codeword $\mathbf{u}^{(0)} = (0, 0, \ldots, 0)$ has been encoded resulting in the transmitted BPSK modulated vector $\ddot{\mathbf{x}}^{(0)} = (+1, +1, \ldots, +1)$, is Gaussian distributed with mean $\mu_E = 2\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$ and variance $\sigma_E^2 = 4\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$. Thus, the bit error probability of the extrinsic output can be determined as

$$\begin{aligned} P_b^{[\text{ext}]}\Big|_{I_A=1} &= \int_{-\infty}^{0} p_E(\xi)\mathrm{d}\xi = \frac{1}{\sqrt{2\pi}\sigma_E}\int_{-\infty}^{0} \mathrm{e}^{-\frac{(\xi-\mu_E)^2}{2\sigma_E^2}}\mathrm{d}\xi\\ &= \frac{1}{2}\,\text{erfc}\left(\frac{\mu_E}{\sqrt{2}\sigma_E}\right) = \frac{1}{2}\,\text{erfc}\left(\frac{\sqrt{\bar{D}_{\mathsf{C}}}}{\sqrt{2}\sigma_n}\right)\,. \end{aligned}$$

■

Corollary 3.4: The mutual information I_E between the data bits $\mathbf{u}$ and the extrinsic output of the decoded bits $\hat{u}$ (L-value representation) of a MAP SISO decoder for a feed forward convolutional code C, given the conditions that perfect *a priori* knowledge on the data bits $\mathbf{u}$ is available ($I_A = 1$ bit), that the data bits $\mathbf{u}$ are equiprobable, and that the transmission is performed on an AWGN channel, depends solely on the noise variance σ_n^2 and on the Hamming weight $\bar{D}_{\mathsf{C}}$ of the impulse response. This mutual information can be expressed as

$$I_E\Big|_{I_A=1} = 1 - \frac{\sigma_n}{\sqrt{8\pi\bar{D}_{\mathsf{C}}}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\left(\xi\sigma_n^2 - 2\bar{D}_{\mathsf{C}}\right)^2}{8\bar{D}_{\mathsf{C}}\sigma_n^2}}\,\mathrm{ld}\left(1+\mathrm{e}^{-\xi}\right)\mathrm{d}\xi\,.$$

Proof: Due to the linearity of convolutional codes, we can assume, without loss of generality, that the all zero sequence has been encoded and thus the $(+1, +1\ldots, +1)$ sequence has been transmitted. Therefore, according to Theorem 3.2, the L-values of the extrinsic decoder output show a Gaussian distribution with mean $\mu_E = 2\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$ and variance $\sigma_E^2 = 4\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$. According to [5], the mutual information I_E can then be expressed as

$$I_E(\sigma_E) = 1 - \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-((\xi-\mu_E)^2/(2\sigma_E^2))}}{\sqrt{2\pi}\sigma_E}\,\mathrm{ld}\left(1+\mathrm{e}^{-\xi}\right)\mathrm{d}\xi. \qquad (6)$$

Substituting $\mu_E = 2\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$ and $\sigma_E^2 = 4\frac{\bar{D}_{\mathsf{C}}}{\sigma_n^2}$ into (6) proves the corollary. ■

IV. Numerical Simulation Results

Figure 5 shows the result of Corollary 3.4, i.e., the mutual information of the extrinsic output of a MAP SISO decoder of a feed forward convolutional code C if perfect *a priori* knowledge is available. It can easily be seen that the upper right point of the EXIT characteristic (i.e., $I_E = 1$ bit, given $I_A = 1$ bit) can closely be reached only for large $\bar{D}_{\mathsf{C}}$ and in good channel conditions. For terminated (or tailbiting) recursive convolutional codes, this plot would be a flat surface, as $\bar{D}_{\mathsf{C}}$ tends to infinity in the case of recursive codes [12].

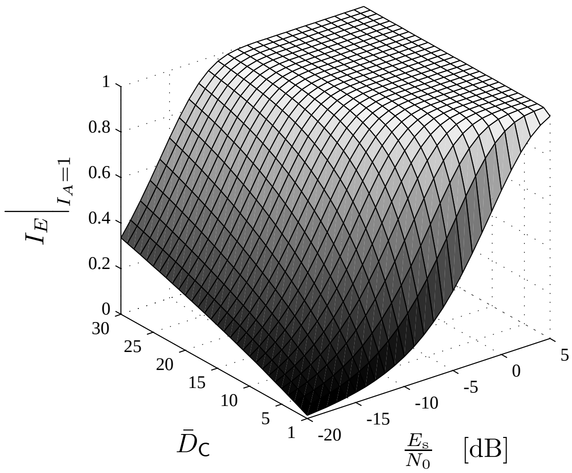

Fig. 5. Mutual information of the extrinsic output as a function of channel quality and $\bar{D}_{\mathsf{C}}$

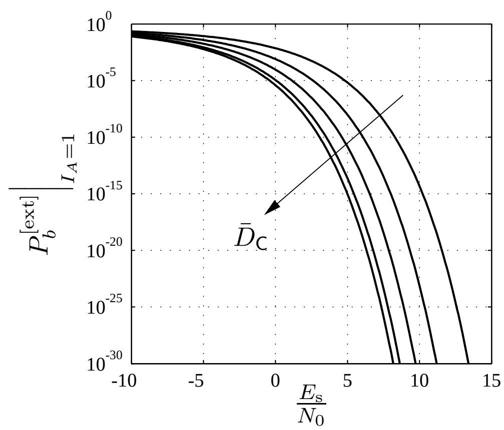

Fig. 6. Hard decision bit error probability of the extrinsic output for $I_A = 1\,\text{bit}$ ($\bar{D}_{\mathsf{C}} = 3, 5, 7, 9, 10$)

Figure 6 shows the hard decision bit error probability of the extrinsic output of the MAP SISO decoder if perfect *a priori* knowledge is available ($I_A = 1\,\text{bit}$). The maximum mutual information of the EXIT characteristics in Fig. 2 (see also Table I) can be read off in Fig. 5 using the information in Table II. Table II also contains the calculated values of $P_b^{[\text{ext}]}$ and $I_E\big|_{I_A=1}$ for a channel quality of $\frac{E_s}{N_0} = -5\,\text{dB}$. It can be seen that the calculated maximum mutual information almost perfectly matches the measured values of Table I. The differences can be explained by numerical inaccuracies during the measurement and/or by the finite histogram resolution.

TABLE II
IMPULSE RESPONSE HAMMING WEIGHTS FOR SOME SELECTED CONVOLUTIONAL CODES AND NUMERICAL RESULTS FOR $E_s/N_0 = 1/(2\sigma_n^2) = -5\,\text{dB}$

Generator polynomials	memory J	$\bar{D}_{\mathsf{C}}$	$P_b^{[\text{ext}]}$	$I_E\big\|_{I_A=1}$
$(3, 2)_8$	1	3	0.0842	0.7038
$(7, 5)_8$	2	5	0.0377	0.8592
$(17, 15)_8$	3	7	0.0177	0.9315
$(17, 15, 13)_8$	3	10	0.0060	0.9762
$(23, 35)_8$	4	7	0.0177	0.9315
$(53, 75)_8$	5	9	0.0085	0.9662
$(133, 171)_8$	6	10	0.0060	0.9762

V. CONCLUSION

In our contribution, we analyzed the behavior of the EXIT characteristics of feed forward convolutional codes. Simulations have shown that the mutual information at the extrinsic output of a MAP SISO decoder for feed forward and non-terminated recursive convolutional codes does not reach $I_E = 1\,\text{bit}$ if perfect *a priori* information is available – a fact that has already been noted in the literature. We give an analytical expression of the attainable mutual information if perfect *a priori* knowledge is available for the case of feed forward convolutional codes. This maximum attainable mutual information solely depends on the channel noise variance and the Hamming weight of the code impulse response. We give an easy explanation of this property using the Trellis representation of the convolutional code. Furthermore, we have shown that the extrinsic output of a MAP SISO decoder is Gaussian distributed if the channel noise is Gaussian, the data bits are equiprobable, and the *a priori* information on the data bits is considered to be perfect. Additionally, we have derived an analytical expression of the mutual information attainable if perfect *a priori* knowledge is available as well as an expression of the hard decision bit error rate of the extrinsic output in this case. The paper concludes with an evaluation of different codes and the verification of the theoretical results.

ACKNOWLEDGMENTS

Parts of this work have been financed by the European Union in the scope of the FP6 research project *FlexCode*.

REFERENCES

[1] C. Berrou and A. Glavieux, “Near Optimum Error Correcting Coding and Decoding: Turbo-Codes,” *IEEE Trans. Comm.*, vol. 44, no. 10, pp. 1261–1271, Oct. 1996.

[2] J. Hagenauer, E. Offer, and L. Papke, “Iterative Decoding of Binary Block and Convolutional Codes,” *IEEE Trans. Inform. Theory*, pp. 429–445, Mar. 1996.

[3] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, “Analysis, Design, and Iterative Decoding of Double Serially Concatenated Codes with Interleavers,” *IEEE J. Select. Areas Commun.*, Feb. 1998.

[4] S. ten Brink, “Convergence of Iterative Decoding,” *IEE Electronics Letters*, vol. 35, no. 10, pp. 806–808, May 1999.

[5] S. ten Brink, “Convergence Behavior of Iteratively Decoded Parallel Concatenated Codes,” *IEEE Trans. Comm.*, pp. 1727–1737, Oct. 2001.

[6] R. Perkert, M. Kaindl, and T. Hindelang, “Iterative Source and Channel Decoding for GSM,” in *Proc. of IEEE ICASSP*, May 2001.

[7] M. Adrat and P. Vary, “Iterative Source-Channel Decoding: Improved System Design Using EXIT Charts,” *EURASIP J. on Appl. Signal Proc. (Special Issue: Turbo Processing)*, no. 6, pp. 928–941, May 2005.

[8] N. Görtz and A. Schaefer, “The Use of EXIT Charts in Iterative Source-Channel Decoding,” in *Proc. of EUSIPCO*, Vienna, Austria, Sept. 2004.

[9] A. Ashikhmin, G. Kramer, and S. ten Brink, “Extrinsic information transfer functions: model and erasure channel properties,” *IEEE Trans. Inform. Theory*, pp. 2657–2673, Nov. 2004.

[10] S. Benedetto, G. Montorsi, D. Divsalar, and F. Pollara, “Soft-Input Soft-Output Modules for the Construction and Distributed Iterative Decoding of Code Networks,” *Europ. Trans. Telecommun. (ETT)*, Mar. 1998.

[11] S. ten Brink, “Exploiting the Chain Rule of Mutual Information for the Design of Iterative Decoding Schemes,” in *Proc. Allerton Conf. on Commun., Control, and Computing*, Monticello, IL, USA, Oct. 2001.

[12] J. Kliewer, A. Huebner, and D. J. Costello, Jr., “On the Achievable Extrinsic Information of Inner Decoders in Serial Concatenation,” in *Proc. of ISIT*, Seattle, WA, USA, July 2006.

[13] A. Papoulis and S. Unnikrishna Pillai, *Probability, Random Variables and Stochastic Processes*. McGraw Hill, 2002.